

A BRIEF SURVEY OF NIGEL KALTON'S WORK ON INTERPOLATION AND RELATED TOPICS

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ABSTRACT. This is the third of a series of papers surveying some small part of the remarkable work of our friend and colleague Nigel Kalton. We have written it as part of a tribute to his memory. It does not contain new results. This time, rather than concentrating on one particular paper, we attempt to give a general overview of Nigel's many contributions to the theory of interpolation of Banach spaces, and also, significantly, quasi-Banach spaces.

The heart of interpolation theory is the problem of constructing *interpolation spaces*. That is, given two Banach (or quasi-Banach) spaces, A_0 and A_1 (compatibly contained of course in some larger space), how can you construct and describe interpolation spaces, A_* , for the pair (A_0, A_1) ? Such spaces A_* should have the interpolation property that a linear operator, T , which is bounded on A_i for $i = 0, 1$ is automatically bounded on this intermediate space A_* . Although that classical question has evolved into a broad multifaceted topic, it remains unified by a few basic themes. Several of those themes were major foci of Nigel Kalton's research and it is no surprise that he made interesting and important contributions to interpolation theory.

We will not give an overview of interpolation theory. Basic books on the topic include [3], [4], [5], [51], [54], [59], as well as some earlier books, for instance [6], and [55], where one can see viewpoints which guided development of the general theory. Furthermore, of course, there is the excellent introduction [48] to the topic written by Nigel¹ himself, with Stephen Montgomery-Smith, which emphasizes the role of interpolation theory in the study of the geometry of Banach spaces, a topic of continuing interest to him.

Nigel's body of work is not a collection of isolated results. It is a richly interconnected web, and tracing individual strands cannot capture its richness. That being said, we will now describe several themes which cover part of Nigel's work on interpolation.

1. THEORY

Some of Nigel's contributions were direct additions to the structural framework of interpolation theory.

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¹The fact that in most places in this survey and also in our previous surveys [15, 16] we allow ourselves the informality of addressing Nigel Kalton by his first name should be understood as part of our expression of our great admiration of his work, of our warm friendship for him, and our deep regret that he is no longer with us.

Many natural questions at the foundation of interpolation theory are now well understood, but several remain recalcitrant. Nigel made important contributions to two of them. One question concerns propagation of compactness; if the operator T , mentioned earlier, is not only bounded on A_i , $i = 0, 1$, but also compact on A_0 ; must it then be compact on the interpolation space A_* ? The answer depends on the details of the method used to construct A_* . For one of the basic classical methods, the complex method, the question remains unanswered. That question, a favorite of one of the current authors, is studied in their joint paper [13] which we hope to briefly discuss in a future survey in this series. Another fundamental question, studied in [38] asks what conditions on (A_0, A_1) insure that the classical real method of interpolation is adequate to construct all possible interpolation spaces associated with the pair (A_0, A_1) . It is classical that the answer is yes for the pair (L^1, L^∞) [8]. (A different way of answering a closely related question can be found in [53].) In [38] Nigel takes our understanding much further for the situation in which L^1 is replaced by a more general rearrangement invariant space.

The paper [14] is joint work with the authors of this survey. We have already discussed it in [16]. It gives an approach to interpolation which unifies the real and complex methods of interpolation (historically the two most important) as well as several of the less well known variants, in a way which enables commutator estimates in the spirit of those studied in [36] to be obtained in a unified way. (See earlier work by Svante Janson [28] for two other ways of unifying a wide range of interpolation methods. We also mention that Gilles Godefroy has discussed the work of [36] (and also several other parts of Nigel's work) in [21] and [22], and we have discussed it in [15].)

Questions involving the functorial properties of interpolation constructions can be quite subtle and results about them can be very useful. For instance, if a certain construction produces an interpolation space A_* from the spaces A_i , $i = 0, 1$, and if the A_i have subspaces B_i , $i = 0, 1$, will it be true that the same construction produces an interpolation space B_* from the B_i 's which has the "expected, natural" relation to A_* ? The general question was studied by Svante Janson in [29]. In particular cases, finding a positive answer to this general question can be exactly the tool needed to solve some other problem. That was done by Nigel in [39] and then again, jointly with Sergei Ivanov in [27].

As more and more techniques and results of classical analysis were extended to quasi-Banach spaces, it became important to have a set of interpolation tools which could work in these new contexts. For instance, in the mid part of the last century, the theory of singular integral operators focused on the operators as acting on the Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$. However, by the latter part of the century the scope of these studies had expanded to include viewing these operators as acting on the Hardy spaces, $H^p(\mathbb{R}^n)$, $0 < p < \infty$; a scale of spaces which coincides with the Lebesgue scale for $p > 1$, which is stable under interpolation, and which, for $p < 1$, consists of quasi-Banach spaces. One of the tools in the extension of the classical results to the range $0 < p \leq 1$ was the theory of complex interpolation of quasi-Banach spaces, including versions of this theory developed by Nigel and several of his coauthors. We will have more to say about this below, in Section 3.

Of course Nigel was a leading expert on quasi-Banach spaces (see [40]) and also studied many other aspects of them which we will not mention here.

2. H^∞ FUNCTIONAL CALCULUS, SECTORIAL OPERATORS

Before the systematic development of approximation theory or semi-group theory or interpolation theory; there was the basic question of how to quantify the degree of smoothness of a function. Counting the number of derivatives which a function has gives a rough, integer based, scale; adequate for many purposes. However both curiosity and necessity rapidly lead to consideration of possible alternatives to, or refinements of, this scale. There are various intuitively attractive ways to do this, but difficulties arise when filling in the technical details, when comparing various approaches to each other, and when exploring the boundaries of applicability of these techniques.

Consider a toy model question; how can we quantify the degree of smoothness of a function in $L^2 = L^2(\mathbb{T}, d\theta)$. We could say that a function in L^2 is “smooth” if the differentiation operator $D = -i\frac{d}{d\theta}$ maps the function into L^2 . The operator D is not bounded but it is closed and densely defined; it is a plausible starting point for a theory. If we want to define a fractional order of smoothness using fractional powers of D we notice that D is not a positive operator and hence it is not clear how we would want to define, say, $D^{1/2}$. A standard next step is to instead work with an associated positive operator of comparable “size”, for instance $|D| = (D^*D)^{1/2}$ or $(I + D^*D)^{1/2}$ which has the additional property, at times very convenient, of being bounded below. These operators are ones for which the classical functional calculus for unbounded operators on Hilbert space works extremely well and we can use that calculus to define the fractional powers. We can then use membership in the domains of definition of the various operators $|D|^\alpha$, $\alpha > 0$ to define a smoothness scale with a continuous parameter. We can easily also prove expected results such as: If $0 < \alpha < \beta$ and if g has smoothness β then $|D|^\alpha g$ has smoothness $\beta - \alpha$.

This analysis can be taken quite far; but it is deceptively easy because passing to Fourier coefficients gives a convenient and explicit diagonalization of D (and therefore also of $|D|$) and the smoothness scale is explicitly defined via simple conditions on Fourier coefficients. However, even while considering this example, it becomes clear that more substantial theoretical tools would be needed in more general situations. What to do if the operator of primary interest is not positive but just has, in some sense, positive real part? In general, what functional calculus can be used to define the fractional powers and show they have the desired computational properties? What can be done if we measure the size of a function with, say, a Banach space norm rather than a Hilbert space norm? Do the domains of fractional powers form an interpolation scale? How do fractional powers of the operator act on domains of other fractional powers? Is it possible to perturb from a detailed analysis of a simple operator and obtain results for a modification of that operator?

The history of research in this area includes work to develop holomorphic functional calculi for various classes of operators on Banach spaces (to, for instance, define fractional powers), work on the theory of operator semigroups (again, think of the fractional powers) and the theory of interpolation of Banach spaces (how are the domains of the fractional powers related?). Recent overviews of the area include [26] and [52]. Many of the most substantial developments in the area were restricted to operators on Hilbert space, for instance [2]. This was in part because, quoting from Nigel's joint paper [43] with Tamara Kucherenko,

“It is an important observation that the theory of sectorial operators on a Hilbert is, in general, simpler and more easily applicable than in general Banach spaces. This is mainly due to the characterization of Hilbert spaces as certain interpolation spaces related to an operator with an H^∞ functional calculus.”

As the quote suggests, one of Nigel’s particular interests in this area was in expanding the theory beyond the restrictions to Hilbert spaces. He did that and more. His contributions include [41], [43], and [44]. An overview of some of his work in the area is in [42].

3. QUASI-BANACH SPACES

Many interesting and useful classes of functions studied in analysis are normed spaces and even Banach spaces. However there are also naturally occurring complete topological vector spaces whose topologies are defined by a functional $\|\cdot\|$ which has the usual properties of a norm, except that it does not satisfy the triangle inequality.

$$(3.1) \quad \|x + y\| \leq \|x\| + \|y\|.$$

A basic example is, for p , $0 < p < 1$, the Lebesgue space $L^p(S, d\mu)$ of μ measurable functions f for which $\int_S |f|^p d\mu < \infty$. The functional $\|f\|_p = (\int_S |f|^p d\mu)^{1/p}$, does not satisfy (3.1), the space is not a Banach space, the Hahn-Banach theorem does not hold, and it is not clear if the space has a non-trivial dual space (the answer depends on μ). Nevertheless $\|\cdot\|_p^p$ does satisfy the triangle inequality

$$(3.2) \quad \|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$$

and hence can be used to define a metric, $d(x, y) = \|x - y\|_p^p$, and L^p is a complete metric space and is an example of a quasi-Banach space. The Hardy spaces, H^p , for $0 < p < 1$, are another family of examples of quasi-Banach spaces.

As we already indicated at the end of Section 1, the basic questions of interpolation theory continue to be natural and interesting for quasi-Banach spaces. However one of the fundamental approaches to interpolation, the complex method of interpolation [7], does not extend comfortably to quasi-Banach spaces. The reason is that at the center of the proofs of the fundamental interpolation inequality is an application of the following inequality, which can be deduced from the maximum principle for Banach space valued holomorphic functions: If f is a function which maps the closed unit disk \mathbb{D} continuously into a Banach space X and is holomorphic in the interior of the disk, then

$$(3.3) \quad \|f(0)\|_X \leq \int_0^{2\pi} \|f(e^{it})\|_X \frac{dt}{2\pi}.$$

This crucial fact fails for quasi-Banach spaces². The quasi-norms of holomorphic functions³ which take values in such a space X need not satisfy (3.3). They can also fail to satisfy a less stringent variant of (3.3), namely

$$(3.4) \quad \|f(0)\|_X \leq C \int_0^{2\pi} \|f(e^{it})\|_X \frac{dt}{2\pi}$$

for some constant $C = C(X)$.

Nigel addressed these matters in an impressive series of papers, some of the later ones written with co-authors including Loukas Grafakos, Svitlana Mayboroda and Marius Mitrea. In the initial papers of this series he built on previous work of a number of mathematicians to carry out an in-depth study of holomorphic functions taking values in quasi-Banach spaces, (See [31] and [32]. Cf. also the work of Philippe Turpin [60].) This enabled him to then proceed and develop a coherent theory of complex interpolation of these kinds of spaces. The paper [35] was his first work in this direction. As its title indicates, Nigel already dealt here with interpolation, not only of pairs of quasi-Banach spaces, but also, more generally, of *families* of such spaces (following [58]), and much of his subsequent work would also apply to families as well as pairs. He identified the condition (3.4) as the fundamental tool that was needed for developing complex interpolation for quasi-Banach spaces (and also studied it for other purposes). He had earlier (in Theorems 3.7 and 4.1 on pp. 305–306 of [32]) characterized the quasi-Banach spaces for which (3.4) holds – their function defining the quasi-norm must be equivalent to a plurisubharmonic function. He and his co-authors also showed that many of the quasi-Banach spaces of classical analysis such as Hardy, Besov, Sobolev, and Triebel-Lizorkin spaces satisfy this condition. (See Section 9 of [46].) In particular, detailed accounts of Nigel and his co-author's version of the theory of complex interpolation can be found in Section 3 of [47] and in Section 7 of [46]. Once the basic tools of the theory were in place, many applications to classical analysis and partial differential equations became possible. Nigel and several of his coauthors presented some

²It was already known, relatively early in the history of this topic, that there exist quasi-Banach spaces for which the maximum principle itself fails, even if we allow a multiplicative constant greater than 1 in the relevant inequality. One explicit example of such a space was given by Aleksandrov [1, Proposition 4.2, p. 49] (Note that Aleksandrov also connects a corollary of this result with some earlier work [30] of Nigel.) Another non-explicit existence proof was obtained, apparently at about the same time, by the anonymous referee of [56] and presented in Section 4 (pp. 260–261) of that paper. The proof there is sufficiently ingenious to make us wonder if that referee could have been Nigel. Who knows? In the light of these results, it is quite intriguing (though apparently not helpful for complex interpolation) that, as shown by Nigel in Section 5 of [32], *every* quasi-Banach space satisfies a slight variant of (3.4), where instead of integration over a circle, the integration is performed over an annulus (which can be arbitrarily thin).

³It is not immediately obvious what is the most appropriate definition of a holomorphic quasi-Banach space valued function. Here we are implicitly using the definition via power series adopted by Nigel and authors of some earlier papers. (See e.g. the first page of [31] for this definition and the reason for using it.) But there is at least one other reasonable alternative definition: When each element of the quasi-Banach space X is itself a complex valued function on some underlying measure space Ω , then a given function $f : \mathbb{D} \rightarrow X$ of the complex variable z can be considered to be holomorphic (in what might be called the “pointwise” sense) if, at almost every $\omega \in \Omega$, the value $f(z, \omega) \in \mathbb{C}$ of that function at ω depends holomorphically on z . Some results about complex interpolation in the context of this alternative “pointwise” definition can be seen, for example, in [25] and the references therein.

of these in [24], [47], [46] and [23]. For example, [46] deals extensively with applications to PDE.

Nigel's and his coauthors' general and systematic approach to complex interpolation of quasi-Banach spaces was preceded by quite a number of earlier papers by many other mathematicians (also including the authors of this survey) proposing various ways of dealing with this challenging topic. Some of these were restricted to particular kinds of spaces, such as H^p spaces. The history of this topic is extensive and complicated and we apologize for not attempting to do it justice here. We can at least mention that (i) on page 3913 of [47] Nigel and Marius briefly describe some of the approaches used in some of these earlier papers, and (ii) on page 157 of [46] Nigel, Svitlana and Marius give several references to work where other versions of complex interpolation have been used for dealing with quasi-normed Hardy spaces and BMO .

4. TRACES AND COMMUTATORS

As the title of this survey indicates, we also wish to briefly discuss some of Nigel's work in some other topics having some connection with interpolation theory. We shall do that in this section. Nigel worked with traces and commutators for most of his career, from his Memoir [33] in 1988 to [45] which he did not live to see in print. His early work in the area was intertwined with his work on interpolation theory, especially the Memoir and the paper [36]. (As already mentioned, surveys of the results of [36] can be found in [21, 22, 15].) In his fundamental paper on traces and commutators, [34], Nigel takes the ideas of those previous works much further. However, being aware that the paper would be of interest to an audience unfamiliar with interpolation theory, he does not emphasize his path through the earlier work.

We will not here review Nigel's extensive work on traces and commutators which includes [34], [33], [19], [18], [45], [49], and [50]. However we will point out a particular set of results which have far flung resonances which we find very intriguing. So, we will tell that story, but extremely informally. Some more or less implicit hints about its connections with interpolation theory may be found in Section 8 of [36] and in the introductory section of [34].

In the mid 20th Century the University of Chicago was a center of research in commutative harmonic analysis; it was the home of Antoni Zygmund and Alberto Calderón, and, for a while, Eli Stein and Charles Fefferman. By the 1950's and 60's much of the classical theory of Fourier analysis had been extended from the L^2 spaces of the line and circle to the corresponding L^p spaces, $1 < p < \infty$, and also to functions of several variables. However there were some fundamental limitations to the theory as it existed then. One problem was the lack of a satisfying endpoint theory at $p = 1$. It was known that in many ways the classical theory failed to extend to the associated L^1 spaces. For functions of one variable, $n = 1$, much of the theory did go through if, instead of working on L^1 , one worked with the Hardy space H^1 , the subspace of L^1 formed by boundary values of functions holomorphic in the upper halfplane. With this insight came new questions, what was the appropriate analog of the Hardy space for functions of n variables and what were the underlying real variable notions that had made the boundary traces of holomorphy so surprisingly useful in one dimension? The theory for the case where the upper halfplane is replaced by the upper

halfspace of \mathbb{R}^{n+1} was thoroughly developed, notably in [20]. Other work in the years that followed led to the introduction of the “real-variable, atomic Hardy space”, a subspace of functions in L^1 which carry the local cancellation properties at the heart of the classical theory. This viewpoint is presented in [12]; [57] gives a more recent picture.

Using the general constructions of [12], one builds inside of $L^1(\mathbb{R}^1)$ the subspace “atomic H^1 ”, which can be denoted by $H_{\text{at}}^1 = H_{\text{at}}^1(\mathbb{R})$ and which is a natural $p = 1$ endpoint for many results. Those constructions are quite general, they work for any metric measure space with certain resemblances to finite dimensional Euclidean space. In particular, associated with $\ell^1 = \ell^1(\mathbb{Z}_{\geq 0})$ there is the atomic Hardy h_{at}^1 . Furthermore, associated with that space is a symmetrized version, h_{sym}^1 which is, roughly, the space of sequences with rearrangements in h_{at}^1 . Precisely, h_{sym}^1 is the space of sequences $a = (a_1, a_2, \dots)$ such that, with $(\tilde{a}_1, \tilde{a}_2, \dots)$ denoting the same sequence of numbers but rearranged so that $|\tilde{a}_n|$ is nonincreasing, we have

$$\|a\| = \sum |a_n| + \sum \left| \frac{\tilde{a}_1 + \tilde{a}_2 + \dots + \tilde{a}_n}{n} \right| < \infty,$$

In fact h_{sym}^1 is a quasi-Banach space with quasi-norm equivalent to $\|\cdot\|$. This space, which is studied in [34], [36], and [33] is, as is discussed in those references, a space which can be naturally viewed as a discrete analog of the space of rearrangements of functions in $H_{\text{at}}^1(\mathbb{R})$. Hence, in contexts where one is measuring the size of sequences using rearrangement invariant functionals, h_{sym}^1 is a candidate to be a useful extension to $p = 1$ of the scale of spaces $\ell^p(\mathbb{Z}_{\geq 0})$, $p > 1$.

The Schatten classes \mathcal{S}_p are the Banach spaces of linear operators on a Hilbert space with the property that their singular values are in the sequence space ℓ^p , $0 < p < \infty$. It was a question of interest in operator theory to know how the Schatten classes, and other analogously defined operator ideals were interrelated under commutation. For instance it was known that, for $p > 1$, we have $\mathcal{S}_p = [\mathcal{S}_{2p}, \mathcal{S}_{2p}]$ where, in the notation of that area $[A, B]$ is the closed linear span of the set commutators $[a, b]$, $a \in A, b \in B$, and this was known that this fails at $p = 1$. For background on this see [17] and [61].

Recalling that the Schatten classes are, in some sense, analogs of the Lebesgue classes, it is perhaps not surprising that a Hardy type space, in this case, h_{sym}^1 , leads to the natural analog for $p = 1$. Let $\mathcal{S}_{h_{\text{sym}}^1}$ be the space of compact operators on a Hilbert space with the property that their sequence of eigenvalues $\{\lambda_n\} \in h_{\text{sym}}^1$. (Note that h_{sym}^1 is rearrangement invariant and hence $\mathcal{S}_{h_{\text{sym}}^1}$ is unitarily invariant. However, in contrast to ℓ^p spaces, h_{sym}^1 is not a solid space. With this in mind the definition of $\mathcal{S}_{h_{\text{sym}}^1}$ is in terms of eigenvalues, not singular values (= approximation numbers.)) In fact in [34] Nigel proves that $\mathcal{S}_{h_{\text{sym}}^1} = [\mathcal{S}_2, \mathcal{S}_2]$. That is

$$A \in \mathcal{S}_{h_{\text{sym}}^1} \text{ if and only if } A = \sum [B_i, C_i]; \quad B_i, C_i \in \mathcal{S}_2,$$

One reason for pointing out this result is to note the analogy with a different result in the theory of Hardy spaces. Now let H^p , $0 < p < \infty$ be the classical Hardy spaces associated to the unit ball \mathbb{B}^n in \mathbb{C}^n . That is, functions in L^p of the surface of \mathbb{B}^n which are boundary values of functions holomorphic in \mathbb{B}^n . The following theorem is from [11]

$$A \in H^1 \text{ if and only if } A = \sum B_i C_i; \quad B_i, C_i \in H^2.$$

The visual analogy between these two results is clear. Some hints at the deeper relations are seen by comparing Nigel's papers which were mentioned with the ideas of [10]. It seems possible that there are more systematic interrelationships to be found.

5. FINALLY

As we already indicated at the outset, it is of course impossible, in the limited framework of this survey, to come anywhere near doing full justice to all of Nigel's broad and extraordinary contribution to interpolation theory. Ideally, many more of Nigel's papers, including ones which we could only mention briefly here, would have been described in more detail in some parallel documents. Among our regrets we note, for example, that the papers [37] and [9], no less fine than those we have mentioned, did not fit comfortably into our narrative. And, given the extent of Nigel's total research output and the interplay of different topics within it, it seems almost certain that we have overlooked other papers of his which are relevant in one way or another to this survey.

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